G-manifolds with positive Ricci curvature and many isolated singular orbits

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Abstract: *We show that in cohomogeneity 3 there are G-manifolds with any given number of isolated singular orbits and an invariant metric of positive Ricci curvature. We show that the corresponding result is also true in cohomogeneity 5 provided the number of singular orbits is even.*

§0 Introduction

The objects under consideration in this paper are compact G-manifolds with finitely many non-principal orbits. Here, G is a compact Lie group acting smoothly and effectively on a smooth compact manifold M. The orbits of such an action are either principal, exceptional (that is, non-principal but having the same dimension as a principal orbit), or singular, meaning that the orbit dimension is strictly lower than that of a principal orbit. The cohomogeneity of such an action is the codimension of a principal orbit, or equivalently the dimension of the space of orbits $G\backslash M$. Note that the union of principal orbits is a dense subset of M.

We will be primarily interested in the invariant geometry of such objects, that is, the geometry of M equipped with a Riemannian metric which is invariant under the G-action.

The motivation for studying manifolds with only finitely many non-principal orbits arose from the study of cohomogeneity one manifolds. Cohomogeneity one manifolds have been studied extensively in recent years, particularly for their geometric properties. (See for example the survey [Z].) Recently, a new example of a positively curved manifold was found among the cohomogeneity one manifolds ([D],[GVZ]), to add to the many known examples of non-negatively curved cohomogeneity-one manifolds (see for example [GZ1]). The underlying philosophy behind these developments is that curvature or other geometric expressions become simpler and more tractable in the prescence of symmetry, as provided by the action of a 'large' Lie group acting isometrically.

Compact cohomogeneity one manifolds belong to one of two types according to the space of orbits. The orbit space could be a circle, in which case all orbits are principal and the manifold is a principal orbit bundle over the circle. The other possiblilty is a closed interval, in which case there are precisely two non-principal orbits corresponding to the end-points of the interval. This is by far the more interesting of the two cases, and provides our main motivating example for studying manifolds with a finite number of non-principal orbits. Such orbits are clearly isolated in the sense that there exist mutually disjoint invariant tubular neighbourhoods about each non-principal orbit.

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G-manifolds with finitely many non-principal orbits in cohomogeneities greater than one have been studied both topologically in [BW1] and geometrically in [BW2]. As it will be relevant later, we will first dicuss briefly the key topological features of these objects.

Let K denote the principal isotropy of the G-action, and H_1,...,H_p denote the non-principal isotropy groups. If N_i denotes a tubular neighbourhood of the non-principal orbit G/H_i, then $M - \cup_{i=1}^p N_i$ has the structure of a principal-orbit bundle. Let B denote the base of this bundle, so $B = G\backslash(M - \cup_{i=1}^p N_i)$. It is clear that B is a manifold with p boundary components. We note that $T_i := \partial N_i$ has two fibration structures: it is fibered by principal orbits, and is also fibered by normal spheres S^r. The isotropy groups H_i act on these normal spheres. If any H_i acts transitively then the cohomogeneity must be one. As we are interested in cohomogeneities greater than one, we have that H_i acts non-transitively, but with only one orbit type. It turns out that such actions are quite tightly constrained (see [B] chapter 4 §6), and this results in the following

Proposition 0.1. *([BW1], Corollary 10). If the cohomogeneity is greater than one, then K is ineffective kernel of the H_i action on S^r, so K is normal in H_i and $H_i/K \cong U(1)$, $N_{SU(2)}U(1)$, $SU(2)$, or is finite, and acts freely and linearly on the normal sphere S^r.*

From this it is easy to deduce:

Corollary 0.2. *([BW1], Corollary 10). If the cohomogeneity is greater than one, then $G\backslash T_i$ is either a complex or quaternionic projective space, or a $\mathbb{Z}_2$ quotient of an odd dimensional complex projective space in the case of a singular orbit, or in the case of an exceptional orbit a real projective or lens space. Also, each $G\backslash N_i$ is a cone over one of these spaces.*

In turn, we obtain the following description of the orbit space structure:

Theorem 0.3. *([BW1], Theorem 3). $G\backslash M$ is the union of a manifold with boundary B, where each boundary component is one of the spaces listed in Corollary 0.2, together with a cone over each boundary component.*

Considering the dimensions of the possible boundary components of B which can arise, we see that if our G-manifold contains a singular orbit, the cohomogeneity must be odd.

It remains to describe the structure of a non-principal orbit neighbourhood. Let L denote one of the groups listed in Proposition 0.1, and let $\alpha : L \to H_i/K$ be an isomorphism. There is a natural action of L on the product $D^{r+1} \times G/K$ (where D^{r+1} denotes a disc), given for $z \in L$ by

$$z(x, gK) \mapsto (zx, g\alpha(z^{-1})K),$$

where the action of L on D^{r+1} is the standard Hopf action on the distance spheres about the origin of the disc. Using the symbol $\times_\alpha$ to denote a quotient under this action, we have

Proposition 0.4. *([BW1], Theorem 3). For a small invariant tubular neighbourhood N of a singular orbit G/H, we have $N \cong D^{r+1} \times_\alpha G/K$. For the tubular neighbourhood boundary T we have $T \cong S^r \times_\alpha G/K$.*

We now turn our attention to the curvature of invariant Riemannian metrics. In the case of cohomogeneity one we have the following result:

Theorem 0.5. *([GZ2]). A compact cohomogeneity one manifold admits an invariant metric with positive Ricci curvature if and only if its fundamental group is finite.*

There is little possibility of proving a result as strong as this in the current context: the space of orbits in cohomogeneity one is either a circle or an interval. Either way, this makes no contribution to the curvature. However, in higher cohomogeneities, it is to be expected that the geometry of the space of orbits will play some role in determining the global geometric properties.

In [BW2] a general existence result is established for positive Ricci curvature metrics on G-manifolds with finitely many non-principal orbits (see Proposition 0.11 below). Using this result, many Ricci positive examples were presented. For instance we have the following collections, which feature the 7-dimensional Aloff-Wallach spaces as singular orbits. The Aloff-Wallach spaces are a 2-parameter family of simply-connected 7-dimensional homogeneous SU(3)-manifolds, which are very important in Riemannian geometry as almost all admit homogeneous metrics with positive sectional curvature (see [Z] page 82). Explicitly, for p_1, p_2 coprime, the Aloff-Wallach space W_{p_1,p_2} is the quotient

$$\mathrm{SU}(3)/\Big\{\mathrm{diag}(z^{p_1}, z^{p_2}, z^{-p_1-p_2}) \,|\, z \in \mathrm{U}(1)\Big\}.$$

In [BW2] Theorems 5 and 6, it is shown that the families in Examples 0.6 and 0.7 below both contain infinitely many homotopy types.

Examples 0.6. *Given any two Aloff-Wallach spaces W_{p_1,p_2} and W_{q_1,q_2}, there is an 11-dimensional $SU(3)$-manifold $M^{11}_{p_1p_2q_1q_2}$ of cohomogeneity three, with orbit space S^3 and two singular orbits equal to the given Aloff-Wallach spaces, which admits an invariant metric of positive Ricci curvature.*

Examples 0.7. *Given Aloff-Wallach spaces W_{p_1,p_2} and W_{q_1,q_2} with $p_1^2 + p_1p_2 + p_2^2 = q_1^2 + q_1q_2 + q_2^2$, there is a 13-dimensional $SU(3)$-manifold $M^{13}_{p_1p_2q_1q_2}$ of cohomogeneity 5, with orbit space a suspension of $\mathbb{C}P^2$ and two singular orbits equal to the given Aloff-Wallach manifolds, which admits an invariant metric of positive Ricci curvature.*

The problem with the Ricci positive examples appearing in [BW2] is that all examples contain either one or two non-principal orbits. A natural question ([BW2] Open Problem number 1) is thus: is it possible to find invariant Ricci positive metrics on manifolds having more than two non-principal orbits? This is an intriguing question as the obvious candidates *just* fail. These candidates are those for which the manifold B is a 3-sphere or a 5-sphere with some discs removed. Thus the boundary components are $\mathbb{C}P^1$s or $\mathbb{H}P^1$s respectively. It is easily checked that conditions (1) and (2) of Proposition 0.11 mean that while two discs can comfortably be removed, taking out three discs results in these conditions just failing to hold.

The main aim of the current paper is to answer the above problem in the affirmative:

Theorem 0.8. *For any given $p \in \mathbb{N}$, there is a cohomogeneity three $SU(3)$-manifold with p isolated singular orbits and an invariant metric of positive Ricci curvature, and a cohomogeneity five $SU(3)$-manifold with $2p$ singular orbits and an invariant metric of positive Ricci curvature.*

Theorem 0.8 follows immediately from the following theorem-examples:

Theorem 0.9. *Given any finite family of Aloff-Wallach spaces $\{W_1^7, ..., W_p^7\}$, there exists an 11-dimensionsal $SU(3)$-manifold of cohomogeneity three with precisely p singular orbits equal to the given Aloff-Wallach spaces, and an invariant metric of positive Ricci curvature.*

Theorem 0.10. *Given any finite family of Aloff-Wallach spaces $\{W_1^7, ..., W_p^7\}$, there exists a 13-dimensionsal $SU(3)$-manifold of cohomogeneity five with precisely two singular orbits equal to each of the given Aloff-Wallach spaces, and an invariant metric of positive Ricci curvature.*

By an obvious adaptation of the proof of Theorem 6 in [BW1], we see that for each p, the families of manifolds in Theorems 0.9 and 0.10 contain infinitely many homotopy types.

The construction behind the examples in Theorems 0.9 and 0.10 (which will be given in §1) relies on two main ingredients, namely Propositions 0.11 and 0.12 below. In Proposition 0.11, g_i denotes a metric on boundary component i of B which is induced via the standard submersion from the round metric of radius one. We adopt the convention that all principal curvatures at a boundary are computed with respect to the inward pointing normal.

Proposition 0.11. *Suppose that $\pi_1(G/K)$ is finite. Then M admits an invariant Ricci positive metric if B admits a Ricci positive metric such that*
1) *the metric on boundary component i is $\lambda_i^2 g_i$, and*
2) *the principal curvatures (with respect to the inward normal) at boundary component i are greater than $-1/\lambda_i$.*

Proposition 0.12. *Let M denote the sphere S^n, $n \geq 3$, from which p small non-intersecting discs have been removed. Then there is a Ricci positive metric on M such that each boundary component is a round sphere of radius $\nu > 0$ and all principal curvatures at the boundary are $> -1/(2\nu)$.*

Proposition 0.11 appeared as Theorem 5 in [BW2]. Proposition 0.12 arises from a construction of Perelman [P]. This construction is subtle and complicated, however the paper [P] presents only an outline. In order to prove Proposition 0.12 we need to carefully establish the ranges and inter-relationships between various constants which are needed in the construction. For this reason, we supply the missing arguments from §3 of [P]. We do this in the next section.

Combining Propositions 0.11 and 0.12 yields:

Corollary 0.13. *Suppose that $\pi_1(G/K)$ is finite, and that the space of orbits B is either S^3 or S^5 with a number of non-intersecting discs removed. Then M admits an invariant metric of positive Ricci curvature.*

Proof. First note that S^3 or S^5 less a number of discs *is* a valid candidate for B, as the boundary components are all equal to $S^2 = \mathbb{C}P^1$ respectively $S^4 = \mathbb{H}P^1$. By Proposition 0.11, we only need to show that B can be equipped with a Ricci positive metric satisfying conditions (1) and (2). By Proposition 0.12 we can equip B in either case with a Ricci positive metric with round boundary components of radius ν with principal curvatures $> -1/(2\nu)$. Now the standard Fubini-Study metric on $\mathbb{C}P^1$ or $\mathbb{H}P^1$ is identical to a round

metric of radius 1/2. Denoting the appropriate Fubini-Study metric by g, we have that a round metric of radius ν is precisely $\lambda^2 g$ with $\lambda = 2\nu$. Thus by Proposition 0.11, M will admit an invariant Ricci positive metric provided the principal curvatures at the boundary are all $> -1/\lambda_i = -1/(2\nu)$, which is true by Proposition 0.12. □

Using Corollary 0.13, we are now in a position to prove Theorems 0.9 and 0.10.

Proof of Theorem 0.9. Given Aloff-Wallach spaces $W_1, ..., W_p$, the SU(3)-manifold in question is just the obvious generalization to p singular orbits of the manifold constructed in the proof of Theorem 5 in [BW1]. As the space of orbits for this manifold is S^3 with p discs removed, the existence of an invariant Ricci positive metric follows immediately from Corollary 0.13.

Proof of Theorem 0.10. Given Aloff-Wallach spaces $W_1, ..., W_p$, consider the SU(3)-manifold M_i^{13} which has two identical singular orbits equal to W_i, as constructed in the proof of Theorem 26 in [BW1]. Away from the singular orbits, the M_i have the structure of SU(3)-bundles. Performing fibre connected sums between the M_i yields an SU(3)-manifold with singular orbits $W_1, W_1, ..., W_p, W_p$ and orbit space an S^5 with $2p$ discs removed. The existance of an invariant Ricci positive metric now follows from Corollary 0.13. □

Theorem 0.8 leaves us with the following natural

Open question. *Can we find manifolds with more than two non-principal orbits and an invariant Ricci positive metric in cohomogeneities* $\neq 3, 5$*?*

A key feature of cohomogeneity 3 and 5 is that if we take any closed manifold of dimension 3 or 5 and remove a disc, the resulting boundary is a projective space, namely $\mathbb{C}P^1$ respectively $\mathbb{H}P^1$. Thus a punctured 3-sphere or a punctured 5-sphere can be taken as the manifold B as in Corollary 0.13. In cohomogeneities $\neq 3, 5$ removing a disc will not produce projective space boundaries. Thus we need to work harder to find candidates for B. For example $\mathbb{H}P^{2k+1}$, $\mathbb{C}P^{2k+1}$ and $\mathbb{R}P^{2k+1}$ are boundaries, and so we can create manifolds with boundary (by a connected sum on the interior of the bounding manifolds) having any selection of these spaces as boundary components. To understand the topology, and especially the geometry of such objects presents a challenge. Although we believe the answer to the above question will be yes, we suspect that constructing examples to show this will prove difficult.

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§1 Proof of Proposition 0.12

Consider a round sphere S^n, $n \geq 3$, and remove a collection of non-intersecting discs. It is easy to check that the resulting boundaries do not satisfy the conclusion of Proposition 0.12. Moreover, it is also intuitively clear that we cannot glue tubes $S^{n-1} \times I$ to the boundary components so as to satisfy the boundary radius and principal curvature

conditions whilst simultaneously maintaining positive Ricci curvature. Perelman's insight in [P] was that the kind of construction we want to achieve in Proposition 0.12 becomes easier if we 'squash' the original sphere.

Consider a warped product metric

$$dt^2 + \cos^2 t \, ds_1^2 + R^2(t) ds_{n-2}^2$$

on S^n, where $t \in [0, \pi/2]$. If $R(t) = \sin t$ then this metric is round of radius 1. The 'squashing' suggested above is based on the singular metric

$$dt^2 + \cos^2 t \, ds_1^2 + R_0^2 \sin^2 t \, ds_{n-2}^2,$$

where $R_0 < 1$ is a constant. Given this metric, the sphere looks like a 'flying saucer', with a singular circle corresponding to $t = 0$. The function $R(t)$ which will be constructed below will take the form $N \sin(t/N)$ for some N in a small neighbourhood of $t = 0$, which ensures that the metric is smooth there. For larger values of t, $R(t)$ will take the form $R_0 \sin t$. Actually, it turns out that we can fix the value $R_0 = 1/10$ in our calculations, and we will do this from now on.

Next, we will remove a number of small geodesic discs of radius r_0 (see Definition 1.7 below) which are centred on the circle $t = 0$. The resulting boundary components are elliptical rather than round. To achieve the round boundaries needed for Proposition 0.12, we have to add tubes $S^{n-1} \times I$ to the boundary components, with the metric on the tubes chosen so as to interpolate from a round boundary component to an elliptical one. Of course, the tube metrics must also give the correct principal curvatures at the 'outer' boundary, and glue smoothly with the punctured sphere at the 'inner' boundary to give a global Ricci positive metric.

The construction of the tube metrics is carried out in §2 of [P], with the result quoted below as Proposition 1.19. The result which allows the smooth Ricci positive gluing between the punctured sphere and the tubes is described in [P] §4, and appears below as Proposition 1.20. In order to establish Proposition 0.12, we need to perform the construction of the punctured sphere metric in detail, and this is now our main task. For the ease of the reader, we have tried to stay as close as possible to Perelman's notation.

We begin with five numerical lemmas, the results of which are needed in the definition of r_0, the radius of the geodesic discs to be removed from S^n.

Lemma 1.1. *There exists a number $c_1 > 0$ such that for all $0 < x < c_1$ we have*

$$\frac{\tan(x^2/2 + x^4/4)}{\tan(x^2/2)} < 1 + \tan^2 x.$$

Proof. It is clear from the Taylor expansion of $\tan x$ that for $0 < x < \pi/2$, $\tan x > x$. Therefore, provided c_1 is sufficiently small, it suffices to show that

$$\tan(x^2/2 + x^4/4) < x^2/x + x^4/2,$$

or equivalently

$$\tan(x^2/2 + x^4/4) - (x^2/x + x^4/2) < x^4/4.$$

Expanding the first term above as a Taylor series, we obtain

$$\frac{1}{3}(x^2/2+x^4/4)^3+\frac{2}{15}(x^2/2+x^4/4)^5+\text{H.O.T.}<x^4/4.$$

As the left-hand-side is $O(x^6)$, the inequality is true for all x sufficiently small. $\square$

Lemma 1.2. *There exists a number $c_2>0$ such that for all $0<x<c_2$ we have*

$$\sin(x^2/2+x^4/4)/x^2<1.$$

Proof. For x sufficiently small we have $\sin(x^2/2+x^4/4)\approx x^2/2$. The result then follows trivially. $\square$

Lemma 1.3. *There exists a number $c_3>0$ such that for all $0<x<c_3$ we have*

$$\sin(x+x^4/4)\cot x<2.$$

Proof. For $0<x<1$ we have $\sin x<x$, and so it suffices to show that

$$x+x^4/4<2\tan x.$$

Since $\tan x>x$ for x small, it suffices to show that

$$x+x^4/4<2x,$$

which is clearly true when x is small. $\square$

Lemma 1.4. *There exists a number $c_4>0$ such that for all $0<x<c_4$ we have*

$$\tan^2 x>\frac{x^2}{2}+\tan^2(\frac{x^2}{2}).$$

Proof. Using the fact that $\tan x>x$ for $0<x<\pi/2$, the inequality will certainly be true if $x^2/2>\tan^2(x^2/2)$. Using the Taylor expansion for $\tan x$ we obtain $\tan^2(x^2/2)=x^4/4+\text{H.O.T.}$ Thus the desired inequality holds for all sufficiently small x. $\square$

Lemma 1.5. *There exists a number $c_5>0$ such that for all $0<x<c_5$ we have*

$$\frac{1}{x^2}\tan(x^2/2)>\frac{1}{40}\Big(1-\frac{x^3}{40}\Big)^{-2}.$$

Proof. As $x\to 0$, the left hand side clearly tends to $1/2$, whereas the right hand side tends to $1/40$. $\square$

Definition 1.6. *Let $\gamma:\mathbb{R}\to\mathbb{R}$ be a standard smooth function with $\gamma'\leq 0$, interpolating between $\gamma(x)=1$ for $x\leq 0$ and $\gamma(x)=0$ for $x\geq\Lambda$, some $\Lambda>10$, such that*

$$\sup_{x\in\mathbb{R}}\{|\gamma^{(k)}(x)|\}<\frac{1}{10}\quad \textit{for } k=1,2.$$

It is clear that such a function γ exists for Λ suitably large.

Definition 1.7. *Let $r_0 > 0$ be any number such that $r_0 \leq \min\{1/\Lambda, c_1, c_2, c_3, c_4, c_5\}$.*

In the next lemma we introduce a function $\mathcal{R}(t)$ which is a first approximation to our desired function $R(t)$. The concavity requirements in (iii) are necessary for curvature considerations. Note that for small t, $\mathcal{R}(t)$ takes the form $N\sin(t/N)$ with $N = r_0/2$, thus ensuring the smoothness of the metric at $t = 0$. For larger t, we would like $\mathcal{R}(t)$ to be $R_0 \sin t$ with $R_0 = 1/10$. However, in order to achieve a C^1-join with the values specified for small t (while maintaining the concavity) there has to be an adjustment, and this is achieved using the function γ. In the following Lemma, note that $r_0 < 1$ and so $r_0^2/2 < r_0$. Thus the adjustment by γ begins at $t \geq r_0 > r_0^2/2$.

Lemma 1.8. *There exists $\eta = \eta(r_0)$ with $0 < \eta < r_0^2/2$ and a C^1-function $\mathcal{R}(t)$ defined for $t \in [0, \pi/2]$ such that*
(i) $\mathcal{R}(t) = (r_0/2)\sin(2t/r_0)$ for $t \in [0, \eta]$;
(ii) $\mathcal{R}(t) = (1/10)\sin(t + (r_0^4/4)\gamma(t/r_0 - 1))$ for $t \in [r_0^2/2, \pi/2]$;
(iii) $\mathcal{R}(t)$ is smooth for $t \in (\eta, r_0^2/2)$ and $-\mathcal{R}''/\mathcal{R} > 2/r_0^2$ for these values of t.

Proof. Since the form of $\mathcal{R}(t)$ is fixed for $t \in [0, \eta]$ and $t \in [r_0^2/2, \pi/2]$, we only have to show how to construct $\mathcal{R}(t)$ in the interval $t \in [\eta, r_0^2/2]$. Suitable values for η will emerge from the construction.

Consider the function $f(t) = c + k(t-\eta)^2$ for $t \geq \eta$, where c and k are *negative* constants. We will model the second derivative of $R(t)$ on such a function. Define a function $G(t)$ by

$$G(t) = \int_\eta^t f(x)\,dx = c(t-\eta) + \frac{1}{3}k(t-\eta)^3.$$

Thus $-G(t)$ measures the area between the graph of $f(t)$ and the t-axis. Finally, set

$$F(t) = \int_\eta^t G(x)\,dx = \frac{1}{2}c(t-\eta)^2 + \frac{1}{12}k(t-\eta)^4.$$

It is easy to see that setting

$$\mathcal{R}(t) = \begin{cases} (r_0/2)\sin(2t/r_0) \text{ for } t \in [0,\eta] \\ F(t) \text{ for } t \in [\eta, r_0^2/2] \\ \frac{1}{10}\sin(t + (r_0^4/4)\gamma(t/r_0 - 1)) \text{ for } t \in [r_0^2/2, \pi/2] \end{cases}$$

gives a C^1-function provided the equations in c and k

$$c(r_0^2/2 - \eta) + \frac{1}{3}k(r_0^2/2 - \eta)^3 = A;$$

$$\frac{1}{2}c(r_0^2/2 - \eta)^2 + \frac{1}{12}k(r_0^2/2 - \eta)^4 = B,$$

are satisfied simultaneously, where

$$A = \frac{1}{10}\cos(r_0^2/2 + r_0^4/4) - \cos(2\eta/r_0)$$

and

$$B = \frac{1}{10}\sin(r_0^2/2 + r_0^4/4) - r_0/2\sin(2\eta/r_0).$$

Note that the constant A (respectively B) is the difference between the derivatives (respectively values) of $(1/10)\sin(t + (r_0^4/4)\gamma(t/r_0 - 1))$ at $t = r_0^2/2$ and $(r_0/2)\sin(2t/r_0)$ at $t = \eta$, and that $A < 0$, $B > 0$. These simultaneous equations have solution

$$c = \frac{A - 4B}{(r_0^2/2 - \eta)(1 - 2(r_0^2/2 - \eta))} \qquad\qquad k = \frac{6}{(r_0^2/2 - \eta)^3}\Big[A - \frac{B}{r_0^2/2 - \eta}\Big].$$

With the above values for c and k, we now check that $F(t)$ satisfies condition (iii) for $t \in (\eta, r_0^2/2)$, that is, $-F''(t) > 2r_0^{-2}F(t)$. Now $F''(t) = f(t) = c + k(t-\eta)^2$, and since c and k are both negative, the smallest value of $-F''(t)$ over the interval $[\eta, r_0^2/2]$ is $-c$. As $F(t)$ is increasing over the interval, the greatest value of $2r_0^{-2}F(t)$ occurs at $t = r_0^2/2$, and is equal to $(1/10)\sin(r_0^2/2 + r_0^4/4)r_0^{-2}$. Thus to show the inequality in (iii), it suffices to show

$$-c > \frac{1}{10}\sin(r_0^2/2 + r_0^4/4)r_0^{-2}.$$

As the denominator in the formula for c is less than 1 (but positive) we can under-estimate the left-hand side of the inequality by $4B - A$. By Lemma 1.2 and the choice of r_0 we can over-estimate the right-hand side by $1/10$. Thus it suffices to show that

$$\frac{2}{5}\sin(r_0^2/2 + r_0^4/4) - 2r_0\sin(2\eta/r_0) - \frac{1}{10}\cos(r_0^2/2 + r_0^4/4) + \cos(2\eta/r_0) > 1/10.$$

In the limit as $\eta \to 0$, the left-hand side tends to $(2/5)\sin(r_0^2/2 + r_0^4/4) - (1/10)\cos(r_0^2/2 + r_0^4/4) + 1$. This quantity is clearly greater than $9/10$, and so the inequality holds for all sufficiently small η. □

Lemma 1.9. *For all $t \in (r_0^2/2, \pi/2]$ we have $\mathcal{R}''(t) < 0$.*

Proof. For these values of t we have

$$\mathcal{R}(t) = (1/10)\sin(t + (r_0^4/4)\gamma(t/r_0 - 1)),$$

and so

$$\begin{aligned}\mathcal{R}''(t) = &-\frac{1}{10}\sin(t + (r_0^4/4)\gamma(t/r_0 - 1))(1 + (r_0^3/4)\gamma'(t/r_0 - 1))^2 \\ &+ \frac{1}{10}\cos(t + (r_0^4/4)\gamma(t/r_0 - 1))(r_0^2/4)\gamma''(t/r_0 - 1).\end{aligned}$$

As $\sup_{x\in\mathbb{R}}\{|\gamma^{(k)}(x)|\} < \frac{1}{10}$ for $k = 1, 2$, we see that

$$\mathcal{R}''(t) < -\frac{1}{10}\sin(t + (r_0^4/4)\gamma(t/r_0 - 1))(1 - (r_0^3/40))^2 + \frac{1}{10}\cos(t + (r_0^4/4)\gamma(t/r_0 - 1))(r_0^2/40).$$

Therefore the result is established if we can show that

$$\tan(t + (r_0^4/4)\gamma(t/r_0 - 1)) > (r_0^2/40)(1 - (r_0^3/40))^{-2}.$$

As $\tan x$ is an increasing function, it suffices to show that

$$\frac{1}{r_0^2}\tan(r_0^2/2) > \frac{1}{40}\Big(1 - \frac{r_0^3}{40}\Big)^{-2},$$

and this is true by Lemma 1.5 and the choice of r_0. □

Lemma 1.10. *There exists* $\epsilon = \epsilon(r_0) > 0$ *such that*

$$1 - \frac{\mathcal{R}'}{\mathcal{R}}(t)\tan t > \epsilon > 0$$

for all $t \in [r_0^2/2, r_0]$.

Proof. For these values of t, $\mathcal{R}(t) = (1/10)\sin(t + (r_0^4/4))$, and therefore

$$\frac{\mathcal{R}'}{\mathcal{R}}(t) = \cot(t + (r_0^4/4)),$$

which is strictly decreasing in t. So

$$\frac{\mathcal{R}'}{\mathcal{R}}(t)\tan t < \cot t \tan t = 1.$$

By the compactness of the interval $[r_0^2/2, r_0]$, the existence of ϵ follows. □

We now show how to smooth $\mathcal{R}(t)$. As $\cot t$ is a strictly decreasing function of t, the following result is clear:

Lemma 1.11. *There exists a number* $\mu_0 = \mu_0(r_0)$ *such that*

(i) $\mu_0 < \eta = \eta(r_0)$;

(ii) $\mu_0 \tan(r_0) < \epsilon = \epsilon(r_0)$, *where* ϵ *is the quantity from Lemma 1.10;*

(iii) for all $t \in [r_0^2/2, r_0]$,

$$\mu_0 < \frac{\cot t - \cot(t + (r_0^4/4))}{1 + \cot t}.$$

Lemma 1.12. *Given any* $\mu \in (0, \mu_0)$, *we can smooth the function* $\mathcal{R}(t)$ *to a function* $R(t)$ *by adjusting the values of* $\mathcal{R}(t)$ *in the intervals* $(\eta - \mu, \eta)$ *and* $(r_0^2/2, r_0^2/2 + \mu)$, *so that*

(a) $-R''/R > 2/r_0^2$ *for all* $t \in [0, r_0^2/2]$;

(b) $-R''/R > 1 - \mu$ *for all* $t \in [r_0^2/2, r_0^2/2 + \mu]$;

(c)

$$\Big|\frac{R'(t)}{R(t)} - \frac{\mathcal{R}'(t)}{\mathcal{R}(t)}\Big| < \mu$$

for $t \in [\eta - \mu, \eta]$ *and* $t \in [r_0^2/2, r_0^2/2 + \mu]$.

Proof. We can smooth $\mathcal{R}(t)$ over the given intervals for arbitrarily small μ keeping both the values of the smoothed function and the values of its first derivative arbitrarily close to the original, and the second derivatives interpolating approximately linearly between the those on either side of the given smoothing intervals. That conditions (a)-(c) can be satisfied by such a smoothing follows easily from the fact that $-\mathcal{R}''/\mathcal{R} = 4/r_0^2$ when $t \in [0, \eta)$, $-\mathcal{R}''/\mathcal{R} > 2/r_0^2$ when $t \in (\eta, r_0^2/2)$, and $-\mathcal{R}''/\mathcal{R} = 1$ for $t > r_0^2/2$. $\square$

Corollary 1.13. *For the smooth function* $R(t)$*, we have* $R''(t) < 0$ *for all* $t \in [0, \pi/2]$*, and* $-R''/R > 1 - \mu$ *for all* $t \in [r_0^2/2, r_0]$.

Proof. The first of these statements follows from Lemma 1.9, and (a) and (b) of Lemma 1.12. The second statement follows from 1.12(b), together with the observation that $-R''/R \equiv 1$ for $t \in (r_0^2/2 + \mu, r_0]$. $\square$

Lemma 1.14. *For all* $t \in [0, r_0]$ *we have*

$$\frac{R'}{R} \tan t \leq 1,$$

with the inequality being strict for $t \in (0, r_0]$.

Proof. Using l'Hôpital's rule we see that

$$\lim_{t \to 0^+} \frac{R'(t)}{R(t)} \tan t = 1.$$

For $t > 0$ we need to check that $R'(t) \sin t < R(t) \cos t$. As these are equal in the limit as $t \to 0^+$, it suffices to compare derivatives, and in particular the result will follow if we can establish $(R'(t) \sin t)' < (R(t) \cos t)'$ for $t > 0$.

Now

$$\begin{aligned} (R'(t) \sin t)' &= \quad R''(t) \sin t + R'(t) \cos t; \\ (R(t) \cos t)' &= - R(t) \sin t + R'(t) \cos t. \end{aligned}$$

For $t \in [0, r_0^2/2]$ we have $R'' << -R$ by 1.12(a), and thus the result follows for these values of t.

For $t \in [r_0^2/2, r_0]$, by 1.10 there exists $\epsilon > 0$ such that

$$1 - \frac{\mathcal{R}'}{\mathcal{R}}(t) \tan t > \epsilon.$$

By 1.12(c) we have

$$\frac{R'}{R}(t) < \frac{\mathcal{R}'}{\mathcal{R}}(t) + \mu.$$

Therefore the result will follow for these values of t if

$$\Big(\frac{\mathcal{R}'}{\mathcal{R}}(t) + \mu\Big) \tan t < 1,$$

that is, if

$$\mu \tan t < 1 - \frac{\mathcal{R}'}{\mathcal{R}}(t) \tan t.$$

Thus it suffices to show that $\mu \tan(r_0) < \epsilon$, and this inequality holds by our choice of μ_0. $\square$

Next, we study principal curvatures at the boundary.

Corollary 1.15. *Equip S^n, $n \geq 3$, with the metric $dt^2 + \cos^2 t\; ds_1^2 + R^2(t) ds_{n-2}^2$ where $t \in [0, \pi/2]$. Remove a ball of radius r_0 centred on the circle $t = 0$. Then the principal curvatures at the resulting boundary are $\geq -\cot r_0$.*

Observation: If $R(t) = \sin t$ for all $t \in [0, \pi/2]$ then the above metric is simply the unit radius round metric, and in this case the principal curvatures would all be identically equal to $-\cot r_0$.

Proof. A straightforward calculation of covariant derivatives shows that the principal curvatures occuring are $-\cot r_0$ and $-(R'(t)/R(t)) \cot r_0 \tan t$. Thus it suffices to show that $(R'(t)/R(t)) \tan t \leq 1$ for $t \leq r_0$, and this is true by Lemma 1.14. $\square$

In the next two lemmas, we investigate the sectional curvature of the intrinsic boundary metrics. Following Perelman, we denote the intrinsic sectional curvature by the symbol K_i. For the rest of the notation, let $T = \partial/\partial t$, and let X denote a vector in the S^1-direction, with $T \wedge X$ denoting the plane spanned by these vectors. We will represent a vector tangent to S^{n-2} by Σ. Let $Y \in T \wedge X$ denote a vector tangent to the boundary. It might be helpful to note that the cosine of the angle between T and the normal vector at any point on the boundary is $\cot r_0 \tan t$. This follows from elementary spherical trigonometry.

Lemma 1.16. *The intrinsic curvatures $K_i(Y \wedge \Sigma)$ satisfy*

$$K_i(Y \wedge \Sigma) > \cot^2 r_0.$$

Proof. By [P] page 162, $K_i(Y \wedge \Sigma)$ is given by the expression

$$K_i(Y \wedge \Sigma) = -\frac{R''}{R}(1 - \cot^2 r_0 \tan^2 t) + \frac{R'}{R} \cot^2 r_0 \tan t (1 + \tan^2 t).$$

This expression is not derived in [P], however it can be obtained by first computing the ambient sectional curvature using the formulas on page 159 of [P] (the latter formulas can themselves be obtained by computing Christoffel symbols, for example), then using the Gauss formula for the sectional curvature of embedded submanifolds (see [doC] page 130), and finally a little spherical trigonometry to obtain the form given above.

Consider $t \in [0, r_0^2/2]$. For t in this range we have $-R''/R > 2/r_0^2$ by Lemma 1.12(a), and thus

$$K_i(Y \wedge \Sigma) > (2/r_0^2)(1 - \cot^2 r_0 \tan^2 t).$$

As $\tan t$ is increasing with t, we see that

$$K_i(Y \wedge \Sigma) > (2/r_0^2)(1 - \cot^2 r_0 \tan^2(r_0^2/2)).$$

Thus to show that $K_i(Y \wedge \Sigma) > \cot^2 r_0$ it suffices to show that

$$(2/r_0^2)(1 - \cot^2 r_0 \tan^2(r_0^2/2)) > \cot^2 r_0.$$

With a little rearrangement, this is equivalent to showing

$$\tan^2 r_0 > r_0^2/2 + \tan^2(r_0^2/2).$$

But this is true by Lemma 1.4 and the choice of r_0.

Claim: For all $t \in [r_0^2/2, r_0]$, $K_i(Y \wedge \Sigma) > \frac{R'}{R}(1 + \cot^2 r_0) \tan t$.

To establish this claim, recall from Corollary 1.13 that $-R''/R > 1 - \mu$ for t in this range. Therefore

$$K_i(Y \wedge \Sigma) > (1 - \mu)(1 - \cot^2 r_0 \tan^2 t) + \frac{R'}{R} \cot^2 r_0 \tan t(1 + \tan^2 t).$$

We therefore need to establish the inequality

$$(1 - \mu) - (1 - \mu) \cot^2 r_0 \tan^2 t + \frac{R'}{R} \cot^2 r_0 \tan t + \frac{R'}{R} \cot^2 r_0 \tan^3 t \geq \frac{R'}{R}(1 + \cot^2 r_0) \tan t.$$

By gathering together the second and fourth terms on the left hand side, moving the third term on the left over to the right, and then simplifying the resulting inequality, we obtain

$$\left[(1 - \mu) - \frac{R'}{R} \tan t\right] \geq \cot^2 r_0 \tan^2 t \left[(1 - \mu) - \frac{R'}{R} \tan t\right].$$

Assuming the term in the square brackets is non-negative, this inequality reduces to $1 \geq \cot^2 r_0 \tan^2 t$, which is true since $t \leq r_0$ by assumption.

To complete the proof of the claim, it remains to show that

$$(1 - \mu) - \frac{R'}{R} \tan t \geq 0.$$

For t in the current range it follows from 1.12(c) that

$$\frac{R'}{R} < \cot(t + r_0^4/4) + \mu.$$

Therefore it suffices to show that

$$1 - \mu - \left[\cot(t + r_0^4/4) + \mu\right] \tan t \geq 0.$$

This rearranges to

$$\mu \leq \frac{\cot t - \cot(t + (r_0^4/4))}{1 + \cot t},$$

and this is true by our choice of μ_0. Thus the claim is established.

To complete the proof of the Lemma, note that we have now established the inequality

$$K_i(Y \wedge \Sigma) \geq \cot(t + r_0^4/4)(1 + \cot^2 r_0) \tan t$$

for $t \in [r_0^2/2, r_0]$. Thus it suffices to show that

$$\cot(t + r_0^4/4)(1 + \cot^2 r_0) \tan t > \cot^2 r_0,$$

which rearranges to

$$\tan(t + r_0^4/4)/ \tan t < 1 + \tan^2 r_0.$$

Computing the derivative of the left hand side shows that this quantity is strictly decreasing if and only if $\sin(2t) < \sin(2(t+r_0^4/4))$, which is true for the values of t under consideration. Thus the maximum of the left hand side for $t \in [r_0^2/2, r_0]$ occurs at $t = r_0^2/2$. Therefore the last inequality is true if it holds at $t = r_0^2/2$. But this follows from Lemma 1.1 and our choice of r_0. □

In the next lemma, Σ_1 and Σ_2 are linearly independent tangent vectors to S^{n-2}.

Lemma 1.17. *The intrinsic curvatures $K_i(\Sigma_1 \wedge \Sigma_2)$ satisfy*

$$K_i(\Sigma_1 \wedge \Sigma_2) > \cot r_0.$$

Proof. Perelman's claim ([P] page 162) that

$$K_i(\Sigma_1 \wedge \Sigma_2) = \frac{1 - R'^2(t)(1 - \cot^2 r_0 \tan^2 t)}{R^2(t)}$$

is easily verified. We show (following Perelman) that

$$\frac{1 - R'^2(t)(1 - \cot^2 r_0 \tan^2 t)}{R^2(t)} \geq \frac{1}{\sin^2 t} - \cot^2 t(1 - \cot^2 r_0 \tan^2 t).$$

The right-hand side of this expression simplifies to $1 + \cot^2 r_0$, which is strictly greater than $\cot^2 r_0$. Thus to establish the Lemma it suffices to establish the above inequality.

Notice that we would obtain equality in the inequality if $R(t) = \sin t$. Notice also that we can bound the left-hand side below by over-estimating both $R(t)$ and $R'(t)/R(t)$. We claim that for all $t \in [0, r_0]$, we have

$$R(t) \leq \sin t \text{ and } R'(t)/R(t) \leq \cot t,$$

where the second of these statements follows immediately from Lemma 1.14. Thus establishing the first claim will complete the proof of the Lemma.

For $t \in [0, \eta - \mu]$ we need to check that $(r_0/2)\sin(2t/r_0) \leq \sin t$. We have equality at $t = 0$, so comparing derivatives it suffices to show that $\cos(2t/r_0) \leq \cos t$, which requires $r_0 \leq 2$, and this is true by the choice of r_0.

For $t \in [\eta - \mu, r_0]$ we begin from the result (1.14) that $R'/R \leq \cos t/\sin t$, or equivalently

$$\frac{d}{dt}\ln R \leq \frac{d}{dt}\ln(\sin t).$$

Integrating, we obtain

$$\ln R(t) - \ln R(\eta - \mu) \leq \ln(\sin t) - \ln(\sin(\eta - \mu)).$$

As $R(\eta - \mu) \leq \sin(\eta - \mu)$ we have

$$\ln R(t) \leq \ln(\sin t) + \ln R(\eta - \mu) - \ln(\sin(\eta - \mu)) \leq \ln(\sin t).$$

Thus $R(t) \leq \sin t$ as required. $\square$

Proposition 1.18. *Let M denote the sphere S^n from which p small, non-intersecting dics have been removed. Then M admits a Ricci positive metric such that all principal curvatures at each boundary component are ≥ -1, the induced metric on each boundary component can be expressed in the form $g = ds^2 + B^2(s)ds^2_{n-2}$ where $s \in [0, \omega\pi]$, $1 > \omega > \tau^{(n-2)/(n-1)}$ with $\tau := \max B(s)$, and g has all sectional curvatures > 1.*

Proof. Begin with the metric $dt^2 + \cos t\ ds_1^2 + R^2(t)ds^2_{n-2}$ on S^n as above. Remove p non-intersecting balls of radius r_0 centered on the circle $t = 0$. Note that we are free to select a smaller value for r_0 (see Definition 1.7) should p be too large for our original choice of r_0. By Lemma 1.15, all principal curvatures at the boundary are $\geq -\cot r_0$. By Lemmas 1.16 and 1.17 the sectional curvatures of the induced boundary metric are all $> \cot^2 r_0$. Therefore, rescaling the metric of M by a factor of $\cot^2 r_0$ produces principal curvatures ≥ -1 and intrinsic sectional curvatures > 1, as required. It is clear that the (rescaled) metric on each boundary component can be expressed as $ds^2 + B^2(s)ds^2_{n-2}$, where the function $B(s)$ satisfies $B(s) = \cot(r_0)R(t(s))$. We therefore have

$$\tau := \max B(s) = \cot(r_0)R(r_0) = (1/10)\cot(r_0)\sin(r_0 + (r_0^4/4)).$$

To find the range of the parameter s, first observe that for the *unrescaled* metric the intrinsic distance between the two 'poles' corresponding to the ends of the range of s is the same as the intrinsic distance between poles if we removed a ball of radius r_0 from the 2-sphere with (round) metric $dt^2 + \cos^2 t ds_1^2$. This distance is $\pi \sin r_0$. Thus for our rescaled metric, this distance is $\pi \cot(r_0)\sin(r_0) = \pi \cos r_0$, and hence the range of the parameter s is $\pi \cos r_0$. Therefore the constant ω in the statement of the Proposition takes the value $\cos r_0 < 1$.

We need to check that $\omega > \tau^{(n-2)/(n-1)}$. Firstly note that by Lemma 1.3 and the choice of r_0, we have $\tau < 1/5$. As $\tau < 1$ we have that $\tau^{(n-2)/(n-1)}$ is decreasing with n. Thus it will suffice to show that $\omega > 1/\sqrt{5}$ as $n \geq 3$, or equivalently $r_0 < \cos^{-1}(1/\sqrt{5})$. This latter quantity is ≈ 1.107, and in particular is greater than 1. As $r_0 < 1$ by definition, we see that the inequality $\omega > \tau^{(n-2)/(n-1)}$ is true.

It remains to show that the metric on M has positive Ricci curvature. This metric is a warped product, and the Ricci curvature formulas for such a metric are well-known (see for example [B] page 266): for the unrescaled metric $dt^2 + \cos^2 t ds_1^2 + R^2(t) ds_{n-2}^2$ we have

$$\begin{aligned}
&\mathrm{Ric}(T,T) = 1 - (n-2)R''/R; \\
&\mathrm{Ric}(X,X) = 1 + (n-2)(R'/R)\tan t; \\
&\mathrm{Ric}(\Sigma,\Sigma) = -R''/R + (R'/R)\tan t + (n-3)(1 - {R'}^2)/R^2; \\
&\mathrm{Ric}(T,X) = \mathrm{Ric}(T,\Sigma) = \mathrm{Ric}(X,\Sigma) = 0.
\end{aligned}$$

Here, X and Σ are as in Lemma 1.16, but this time we assume in addition that both are *unit* vectors. Since $-R''(t)/R > 0$ by Corollary 1.13 and $0 \le R' \le 1$, we see immediately that this metric has positive Ricci curvature. Rescaling the metric simply rescales the Ricci curvature, and so has no effect on the positivity. □

The next result appears on page 159 of [P], and is the conclusion of §2 of that paper.

Proposition 1.19. *Let g be a rotationally symmetric metric on S^{n-1} with sectional curvature > 1, distance between the poles $\pi\omega$ and waist $2\pi\tau$; that is, g can be expressed as $ds^2 + B^2(s)ds_{n-2}^2$, where $s \in [0, \pi\omega]$ and $\max B(s) = \tau$. Suppose that $\omega > \tau^{(n-2)/(n-1)}$, and let $\rho \in (\tau^{(n-2)/(n-1)}, \omega)$. Then there exists a metric of positive Ricci curvature on $S^{n-1} \times [0,1]$ such that (a) the boundary component $S^{n-1} \times \{1\}$ has intrinsic metric g and is strictly convex with all principal curvatures > 1; (b) the boundary component $S^{n-1} \times \{0\}$ is concave with all principal curvatures equal to $-\lambda$ and is isometric to a round sphere of radius ρ/λ, for some $\lambda > 0$.*

The idea is to glue a tube as described in Proposition 1.19 onto each of the boundary components of M. To do this, we need the following gluing result, which is the conclusion of §4 in [P]:

Proposition 1.20. *Suppose that N_1 and N_2 are compact smooth Riemannian manifolds with positive Ricci curvature and isometric boundaries. If the principal curvatures at ∂N_1 are strictly greater than the negatives of the corresponding principal curvatures at ∂N_2, then the union $N_1 \cup N_2$ can be smoothed in a small neighbourhood of the gluing to produce a manifold of positive Ricci curvature.*

Combining Propositions 1.18, 1.19 and 1.20 in the obvious way, we arrive at the following:

Corollary 1.21. *For any choice of $\rho \in (\tau^{(n-2)/(n-1)}, \omega)$, the manifold M admits a Ricci positive metric such that each boundary component is a round sphere of radius ρ/λ with all principal curvatures equal to $-\lambda$.*

Proof of Proposition 0.12. We need to show that we can choose ρ in Corollary 1.21 so that the boundary metrics are round of radius ν and the principal curvatures at the boundary are $> -1/(2\nu)$. From Corollary 1.21 we have $\nu = \rho/\lambda$, and the principal curvatures all equal to $-\lambda$. Thus the Proposition will be proved provided $-\lambda > -\lambda/2\rho$, that is, provided $\rho < 1/2$. Now $\rho \in (\tau^{(n-2)/(n-1)}, \omega)$. We have $\omega = \cos r_0$, so this upper bound does not

force $\rho < 1/2$. As ρ can be taken to be any value in this interval, it therefore suffices to show that the lower bound $\tau^{(n-2)/(n-1)} < 1/2$. However, in the proof of Proposition 1.18 we argued that $\tau < 1/\sqrt{5}$. As $\sqrt{5} > 2$ the result follows. $\square$

References

[B] A.L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin (2002).

[BW1] S. Bechtluft-Sachs, D. J. Wraith, *On the topology of G-manifolds with finitely many non-principal orbits*, Topol. Appl. **159** (2012), 3282-3293.

[BW2] S. Bechtluft-Sachs, D. J. Wraith, *On the curvature of G-manifolds with finitely many non-principal orbits*, Geom. Dedicata (to appear).

[D] O. Dearricott, *A 7-manifold with positive curvature*, Duke Math. J. **158** no. 2 (2011), 307-346.

[doC] M. do Carmo, *Riemannian Geometry*, Birkhäuser, (1992).

[GVZ] K. Grove, L. Verdiani, W. Ziller, *An exotic T_1S^4 with positive curvature*, GAFA **21** no. 3 (2011), 499-524.

[GZ1] K. Grove, W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. (2) **152** no. 1 (2000), 331-367.

[GZ2] K. Grove, W. Ziller, *Cohomogeneity one manifolds with positive Ricci curvature*, Invent. Math. **149** (2002), 619-646.

[P] G. Perelman, *Construction of manifolds of positive Ricci curvature with big volume growth and large Betti numbers*, 'Comparison Geometry', Cambridge University Press, (1997).

[Z] W. Ziller, *Examples of manifolds with non-negative sectional curvature*, Surveys in Differential Geometry volume XI, International Press (2007).

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